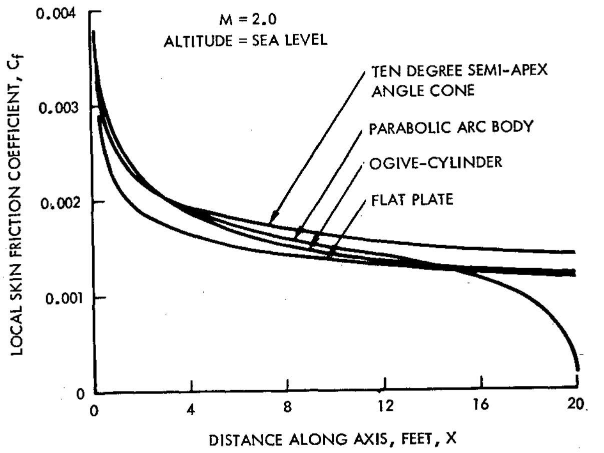

Fig. 3 Body shape effects on local skin-friction distribution.

ment is obtained throughout the supersonic range of $M = 1.0$ to 4.0. Figure 2 shows the effect of varying the over-all fineness ratio for ogive-cylinder combinations at freestream Mach numbers from 1.5 to 3.0.

The effect of ogive-to-cylinder length ratio is small as compared to the effect of the over-all fineness ratio, which affects average skin friction greatly below an over-all fineness ratio of approximately 7. Figure 3 compares the effects of various body shapes on local skin-friction coefficient, for a typical freestream Mach number of 2.0. The ogive cylinder, cone, and parabolic body of revolution show somewhat the same general skin-friction distribution along their forebodies where pressure gradients are favorable ($dp/dx < 0$). The effects of the adverse pressure gradients on the local skin-friction distribution are most pronounced on the parabolic body of revolution which has a rapidly increasing pressure near the aft end. This creates a rapidly falling skin-friction coefficient that approaches the point of separation. The calculation procedure appears to be capable of estimating the separation point, in the case of adverse pressure gradient, although no attempt to verify its accuracy is made.

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Stability of Finite-Difference Equation for the Transient Response of a Flat Plate

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In the course of investigating the transient response of thin shells according to methods outlined by Pian[1] and Leech,[2] the author was led to consider certain finite-difference forms of the equation of motion which governs the transient response of a flat plate. In particular, the problem arose as to what the limits on the ratio of the time mesh to the space mesh might be for an explicit finite-difference formula that would be useful for a numerical calculation of the transient response of such plates.

Fortunately, von Neumann's method of stability analysis as outlined by O'Brien, Hyman, and Kaplan[3] is applicable to this problem, and this analysis (without explanation) is applied herein to the problem at hand.

If one lets D be the customary flexural rigidity of the plate and m be the mass per unit area, the homogeneous plate equation may be written

$$D\nabla^4 w + m\ddot{w} = 0 \tag{1}$$

where w is the displacement normal to the initial plane of the plate. An explicit finite-difference equation, which is equivalent to Eq. (1), may be written as follows,[4] if a "square" mesh $\Delta x = \Delta y$ is used:

$$\begin{aligned}[D/(\Delta x)^4]\,\{&w_{j-2,\,k} - 8w_{j-1,\,k} + 20w_{j,\,k} - 8w_{j+1,\,k} + \\ &w_{j+2,\,k} + 2w_{j-1,\,k+1} - 8w_{j,\,k+1} + 2w_{j+1,\,k+1} + w_{j,\,k+2} + \\ &2w_{j-1,\,k-1} - 8w_{j,\,k-1} + 2w_{j+1,\,k-1} + w_{j,\,k-2}\}_n + \\ &[m/(\Delta t)^2]\{w_{n+1} - 2w_n + w_{n-1}\}_{j,\,k} = 0\end{aligned} \tag{2}$$

In the preceding equation, j and k denote space mesh stations x and y, respectively, and n denotes the instant of time t. Note that Eq. (2) may be used to calculate $w_{n+1,j,\,k}$ when $w_{n,j,\,k}$ and $w_{n-1,\,j,\,k}$ are known.

At least two types of error are associated with such finite-difference equations. The first, truncation error, arises because of the finite distance between points of the finite-difference mesh. This error will not be discussed here since it will be assumed that one is satisfied with the finite-difference approximation. Here then, one is concerned with the second kind of error which is usually considered to be that due to round off associated with using only a finite number of significant figures for any one calculation step; this type of error may grow with increasing time in the stepwise calculation and, if so, would render the numerical solution inaccurate. One wishes to investigate the circumstances under which the error will not grow with time, but instead will die out and thus provide an acceptable solution. This may be regarded as the problem of stability of the finite-difference calculation.

It may be shown[3] that this round off error $\delta(x, y, t)$ must satisfy a similar equation, namely

$$\begin{aligned}[D/(\Delta x)^4]\{&\delta_{j-2,\,k} - 8\delta_{j-1,\,k} + 20\delta_{j,\,k} - 8\delta_{j+1,\,k} + \\ &\delta_{j+2,\,k} + 2\delta_{j-1,\,k+1} - 8\delta_{j,\,k+1} + 2\delta_{j+1,\,k+1} + \\ &\delta_{j,\,k+2} + 2\delta_{j-1,\,k-1} - 8\delta_{j,\,k-1} + 2\delta_{j+1,\,k-1} + \\ &\delta_{j,\,k-2}\}_n + [m/(\Delta t)^2]\{\delta_{n+1} - 2\delta_n + \delta_{n-1}\}_{j,\,k} = 0\end{aligned} \tag{3}$$

Formulas for the difference equivalent to the biharmonic operator, used previously, are given by Crandall.[4] Let it be

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assumed that the general term for the error may be expressed in the form

$$\delta(x, y, t) = e^{\alpha t} e^{i\beta x} e^{i\gamma y} \tag{4}$$

or equivalently

$$\delta_{j,k,n} = e^{\alpha n\Delta t} e^{i\beta j\Delta x} e^{i\gamma k\Delta y} \tag{5}$$

If one lets $\xi = e^{\alpha\Delta t}$ and $r = \Delta t/(\Delta x)^2$, Eqs. (5) and (3) may be combined to give

$$\xi^2 - 2A\xi + 1 = 0 \tag{6}$$

where

$$A = 1 - \frac{8Dr^2}{m}\left[\sin^2\frac{\beta\Delta x}{2} + \sin^2\frac{\gamma\Delta y}{2}\right]^2 \tag{7}$$

It can be shown that the error will not grow with increasing time as long as the following necessary and sufficient condition is applied

$$|\xi| \leq 1 \tag{8}$$

In terms of A, this requirement becomes

$$-2 \leq -\frac{8Dr^2}{m}\left[\sin^2\frac{\beta\Delta x}{2} + \sin^2\frac{\gamma\Delta y}{2}\right]^2 \leq 0 \tag{9}$$

The right-hand inequality is satisfied for all values of r, but the left-hand inequality imposes the following restriction on r

$$r \leq \tfrac{1}{4}(m/D)^{1/2} \tag{10}$$

By satisfying this condition, the calculation of the transient response of an impulsively-loaded flat plate, for example, is assured to be stable.

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Nearly Self-Similar Unsteady Motion

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THE self-similar motion associated with a very strong shock in one-dimensional unsteady flow or the analogous, hypersonic small-disturbance flow has been studied by many investigators. A review and list of references can be found in an article by Mirels.[1] As an extension, he also treated the problems of perturbation on self-similar solutions in hypersonic small-disturbance flow.[2] The present study considers unsteady motions that approach self-similarity asymptotically at large time. The strong-shock assumption is made, and the problem is attacked by using asymptotic expansions in large time such that the leading approximation is a self-similar solution. Although the study ultimately aims toward determining the entire asymptotic flow field,† the purpose of this brief note is to emphasize and study the following specific considerations.

1. Nonuniformity of Approximation

The perturbation equations governing the second term of the asymptotic expansions are singular at the face of the piston. In consequence, some of the assumed asymptotic expansions are not valid in this neighborhood; however, several methods may be used to render the approximation uniformly valid. One is the method of matched asymptotic expansions with one set of expansions valid in an "*outer*" region *away* from the piston and another set valid in an "*inner*" region *near* the piston.[3] In Mirels' treatment of nearly self-similar motion, this nonuniformity is not mentioned and his solution corresponds only to the outer expansions of the present formulation.

2. Possible Disparity in Perturbation Powers

Although it seems natural to assume the same power-law perturbation for both the body and shock wave, the possibility of a disparity in the perturbation powers cannot be ruled out. In fact, the recent studies of Guiraud[4] and Messiter[5] of a special power-law perturbation, that is a result of the displacement effect of an entropy layer, to a particular self-similar solution (the blast wave) provide an example of such disparity. (Vaglio-Laurin's different result[6] on the same problem is due to an error made in his matching process.[7]) In Mirels' work, the cases with disparity in the perturbation power are implicitly excluded from consideration.‡

Formulation of the Problem

Consider the motion of the piston to have an asymptotic behavior at large time

$$x_b = t^m(\eta_{b0} + 1/t^N + \ldots) \tag{1}$$

where x and t are nondimensional distance and time, respectively. A constant η_{b0} is to be determined later, and m and N are positive numbers§ with $2/(j+3) < m < 1$ and $j = 0$, 1, and 2 for plane, cylindrical and spherical motion, respectively. The associated motion of a shock wave can be assumed to have an asymptotic behavior at large time

$$x_s = t^m(1 + a_k/t^{kN} + \ldots) \tag{2}$$

where the constants k and a_k are to be determined. Although intuition indicates $k = 1$, it will be established that cases exist in which $k \neq 1$. The asymptotic expansions for the nondimensional velocity v, pressure p, and density ρ will be assumed according to their form behind the shock wave, which is considered to be very strong. Let

$$\left.\begin{aligned} \eta &= x/t^m \\ v &= mt^{m-1}[V_0(\eta) + V_k(\eta)/t^{kN} + \ldots] \\ p &= m^2t^{2(m-1)}\left[P_0(\eta) + \frac{P_k(\eta)}{t^{kN}} + \ldots\right] \\ \rho &= D_0(\eta) + \frac{D_k(\eta)}{t^{kN}} + \ldots \end{aligned}\right\} \tag{3}$$

Substitution of Eqs. (3) into the conservation equations, the strong shock conditions and the boundary condition $v_b =$

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† The analysis of this problem will be presented more fully in a subsequent paper in collaboration with H. K. Cheng.

‡ The author is informed by H. K. Cheng that recently Mirels independently observed the possibility of disparity in the perturbation powers.

§ The analysis holds for small instead of large time, if N is a negative number.